Last time: Cross Product 9/3

Ex: $\vec{u} = \langle 7,-1,3 \rangle$, $\vec{v} = \langle -4,9,6 \rangle$

$$\vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 7 & -1 & 3 \\ -4 & 9 & 6 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 3 \\ 9 & 6 \end{vmatrix}\hat{i} - \begin{vmatrix} 7 & 3 \\ -4 & 6 \end{vmatrix}\hat{j} + \begin{vmatrix} 7 & -1 \\ -4 & 9 \end{vmatrix}\hat{k} \quad \Big\}\ \text{Symbolic Determinant } (\hat{i},\hat{j},\hat{k})$$

$$= \big((-1)(6)-(3)(9)\big)\hat{i} - \big((7)(6)-(3)(-4)\big)\hat{j} + \big((7)(9)-(-1)(-4)\big)\hat{k}$$

$$= (-33)\hat{i} - 54\hat{j} + 59\hat{k}$$

$$= \langle -33, -54, 59 \rangle$$

Recall: Prop (Properties of Cross Product):

Let $\vec{u},\vec{v},\vec{w} \in \mathbb{R}^3$ and $c \in \mathbb{R}$

Algebraic Properties:

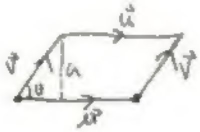
1. $\vec{u}\times\vec{v} = -(\vec{v}\times\vec{u})$
2. $(c\vec{u})\times\vec{v} = c(\vec{u}\times\vec{v}) = \vec{u}\times(c\vec{v})$
3. $\vec{u}\times(\vec{v}+\vec{w}) = \vec{u}\times\vec{v} + \vec{u}\times\vec{w}$
4. $(\vec{u}+\vec{v})\times\vec{w} = \vec{u}\times\vec{w} + \vec{v}\times\vec{w}$
5. $\vec{u}\cdot(\vec{v}\times\vec{w}) = (\vec{u}\times\vec{v})\cdot\vec{w}$
6. $\vec{u}\times(\vec{v}\times\vec{w}) = (\vec{u}\cdot\vec{w})\vec{v} - (\vec{u}\cdot\vec{v})\vec{w}$

Geometric Property:

7. $\vec{u}\times\vec{v}$ is orthogonal to $\vec{u}$ and $\vec{v}$
8. $|\vec{u}\times\vec{v}| = |\vec{u}|\cdot|\vec{v}|\sin(\theta)$, for θ, the angle between $\vec{u}$ and $\vec{v}$
9. $\vec{u}\times\vec{v} = \vec{0}$ if and only if (iff) $\vec{u}$ and $\vec{v}$ are $\parallel$.

NB: Cross product obeys "right-hand rule." (Direction)

- As for the magnitude.



$\sin\theta = \frac{a}{|\vec{v}|}$ $\quad a = |\vec{v}|\sin\theta$

$\therefore$ Area of parallelogram is

$A = (\text{altitude})(\text{base})$

$= a|\vec{u}| = |\vec{u}||\vec{v}|\sin\theta$

Point: if we know ⑧, we know $A_{\text{parallelogram}} =$ magnitude of the cross product.

Proof of part 8 of the proposition:

We compute as the following:

$|\vec{u}\times\vec{v}|^2 = (\vec{u}\times\vec{v})\cdot\underbrace{(\vec{u}\times\vec{v})}_{\vec{w}}$ property of dot product

$= \vec{u}\cdot(\vec{v}\times(\vec{u}\times\vec{v}))$ property 5 of cross

$= \vec{u}\cdot((\vec{v}\cdot\vec{v})\vec{u} - (\vec{v}\cdot\vec{u})\vec{v})$ property 6 of cross

$\left.\begin{aligned} &= \vec{u}\cdot((\vec{v}\cdot\vec{v})\vec{u}) - \vec{u}\cdot((\vec{v}\cdot\vec{u})\vec{v}) \\ &= (\vec{v}\cdot\vec{v})(\vec{u}\cdot\vec{u}) - (\vec{v}\cdot\vec{u})(\vec{u}\cdot\vec{v}) \\ &= |\vec{v}|^2|\vec{u}|^2 - (\vec{u}\cdot\vec{v})^2 \end{aligned}\right\}$ Properties of dot

$= |\vec{u}|^2\cdot|\vec{v}|^2 - (|\vec{u}||\vec{v}|\cos\theta)^2$ geometric interpretation of dot product

$= |\vec{u}|^2\cdot|\vec{v}|^2 - |\vec{u}|^2\cdot|\vec{v}|^2\cos^2\theta$

$= |\vec{u}|^2|\vec{v}|^2(1-\cos^2\theta)$

$= |\vec{u}|^2|\vec{v}|^2\sin^2\theta$

$= (|\vec{u}|\cdot|\vec{v}|\sin\theta)^2$

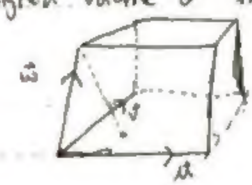
$\therefore$ So, $|\vec{u}\times\vec{v}|^2 = (|\vec{u}|\cdot|\vec{v}|\sin\theta)^2$

On the other hand, θ is the geometric angle between $\vec{u}$ and $\vec{v}$.

$\therefore \theta\in[0,\pi]$,

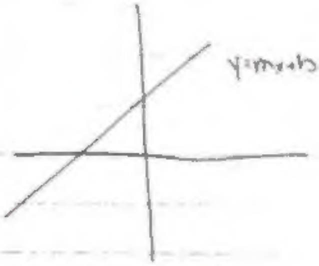
So, $\sin\theta \geq 0$. Hence $\therefore |\vec{u}|\cdot|\vec{v}|\sin\theta = |\vec{u}\times\vec{v}|$

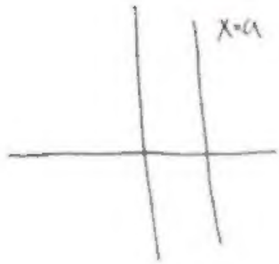
Cor: The scalar triple product $\vec{u}\cdot(\vec{v}\times\vec{w})$ computes the signed volume of the parallelepiped determined by $\vec{u},\vec{v},\vec{w}$.



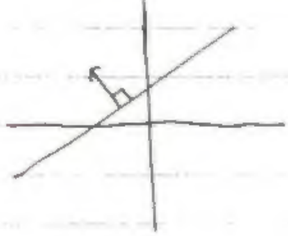
* proof in video on website

12.5: Lines & Planes





$ax+by=C$ better line equation



looks like $\vec{n}\cdot\langle x,y\rangle = C$

in 3-Space, examine $\vec{n}\cdot\vec{x} = d$ $\quad(\vec{n}\neq\vec{0})$

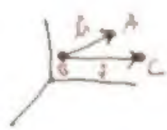
i.e. $\langle a,b,c\rangle\cdot\langle x,y,z\rangle = d$ → vector equation

i.e. $ax+by+cz=d$

This is a plane in 3-Space

Note: given 2 vectors both non-parallel, we get a plane.
One normal vector to that plane is the cross product of the vectors.

Ex: Compute an equation of the plane containing the points:
$(0,1,3)$, $(2,4,0)$, and $(1,2,3)$



Soln: Note that the vectors

$\vec{u} = \langle 2-0, 4-1, 0-3\rangle = \langle 2,3,-3\rangle$

$\vec{v} = \langle 1-0, 2-1, 3-3\rangle = \langle 1,1,0\rangle$

$\vec{u}\times\vec{v} = \vec{n}$ computes the normal vector.

$$\vec{n} = \vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -3 \\ 1 & 1 & 0 \end{vmatrix} = \langle 3,-3,-1\rangle$$

∴ The plane has eq: $\vec{n}\times\vec{x} = d$ i.e. $3x-3y-z=d$

∴ using $(0,1,3)$, $d = 3(0)-3(1)-3 = -6$

→ Plane Eq: $3x-3y-z=-6$